Approximate Solutions to Retirement Spending Problems and The Optimality of Ruin

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Abstract

Milevsky and Huang (2011) investigated the optimal retirement spending policy for a utility-maximizing retiree facing a stochastic lifetime but assuming deterministic investment returns. They solved the problem using techniques from the calculus of variations and derived analytic expressions for the optimal spending rate and wealth depletion time under the Gompertz law of mortality. Of course, in the real world financial returns are stochastic as well as lifetimes, raising the question of whether their qualitative insights and approximations are generalizable or practical.

We solve the retirement income problem when investment returns are indeed stochastic using numerical PDE methods, assuming the principles of stochastic control theory and dynamic programming. But then – and this is key – we compare the proper optimal spending rates to the analytic approach presented in Milevsky and Huang (2011) by *updating* the portfolio wealth inputs to current market values. Our main practical conclusion is that this simplistic approximation when calibrated properly and frequently can indeed be used as an accurate guide for rational retirement spending policy.

As a by-product of our PDE-based methodology, our results indicate that even though the wealth depletion time is no longer a certainty under stochastic returns, the expected age at which liquid wealth is exhausted (i.) takes place well before the maximum lifetime and (ii.) is also well approximated by our analytical solution.

1 Introduction and Motivation

With the continued global decline in Defined Benefit (DB) pension provision and the increased reliance on Defined Contribution (DC) investment plans, researchers and practitioners continue to focus on the proper tools and strategies needed to finance retirement income.

Indeed, the conventional wisdom and practice in the wealth management industry is that individuals should manage their financial affairs – and in particular select a spending or drawdown rate[1] – that reduces the risk of running out of money during retirement. In other words, the goal or objective is to identify and then stick to a so-called safe withdrawal rate that minimizes the risk of ruin. This laudable and rather intuitive objective function has been at the core of hundreds of scholarly as well as practitioner papers written during the last 25 years. They all implicitly or explicitly derive spending plans that seek to avoid (at all costs) the chances of the financial portfolio hitting the value of zero during their life. After all, nobody wants to "go broke" and starve to death. See the work by Bengen (1994); Milevsky and Robinson (2005); Pfau and Kitces (2014) for examples of this paradigm at work. In particular, Bayraktar and Young (2007) show that when utility is a power function and the consumption rate is proportional to wealth, the individual who minimizes lifetime ruin probability behaves like an individual who maximizes the expected discounted utility of consumption.

As a by-product and result of our (more) robust optimization model, we are able to strengthen the claim that within the context of a rational lifecycle model it might be optimal to spend-down financial investment accounts to zero. In other words, from a lifecycle perspective, ruin is not a scenario or outcome that should be avoided at all costs. Rather, the rational objective should be to slowly and smoothly deplete financial resources accounting for the declining probabilities of living to very old ages. And, if a by-product of this behavior is that financial wealth is expected to hit zero at some distant point, so be it – provided there is some pension income to fall back on. In other words, ruin should not be feared if annuities are part of the retiree's portfolio. We provide the stochastic model to justify this claim.

Now, we should note that we are not the first, nor the only authors to point this out. The earliest theoretical hint that this might be the case was in a technical article by Leung (1994) who extended the

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[1]Spending Rate or Withdrawal Rate is defined as the amount of money withdrawn per \$100 from investible wealth. When pensions are present, we deduct the pensions from consumption, thus mathematically, spending rate (per \$100) is $c_t - \pi$, where c_t is consumption and π is pension.

landmark paper by Yaari (1965). This fact was also mentioned (tangentially) in a more practitioner-oriented paper: Milevsky and Huang (2011). However, these articles were structured in an economic environment with deterministic investment returns and no background uncertainty other than mortality.

In sum, in addition to solving a particular stochastic optimization problem, this brief paper reinforces three points that are relevant to retirees and their financial advisors: (1.) Based on our model for optimal consumption, deterministic approximations to stochastic problems provide reasonably accurate rules of thumb. (2.) Risk aversion and one's personal attitude to lifetime uncertainty has a significant impact on the optimal spending rate at retirement. (3.) When one has access to some sort of pension annuity income (i.e. social security income, DB pension, or income annuities), running out of money during retirement shouldn't necessarily be feared or avoided. It might in fact be optimal.

1.1 Literature Review

The topic of sustainable withdrawal rates from a portfolio comprising cash, fixed income, and equity has been investigated quite extensively. The literature on portfolio withdrawal rates can be divided into two categories: one that uses a ruin probability minimization approach to determine a sustainable withdrawal rate or a life-cycle approach based on a utility model that takes investor risk preferences into consideration.

Famous among the practitioners is Bengen (1994) that concluded, empirically by using historical returns of the equity and fixed income markets, that an initial rate of 4% adjusted for inflation annually would sustain the portfolio for 30 years. It has now become the de facto rule of thumb and is commonly referred to as the Bengen 4% rule in the industry. Other studies on a sustainable withdrawal rate include Cooley et al. (1998), often referred to as the Trinity Study. Many of these studies are based on empirical evidence and generally lack the economic framework — a concern pointed out by Scott et al. (2008) which alluded to the need for a life-cycle approach. Gerrard et al. (2006) take a risk management approach whereby they tackle the problem using optimal control theory and determine how a member of a DC pension plan can make spending decisions before mandatory annuitization of the remaining wealth at 75. Stout and Mitchell (2006) develop a dynamic model of retirement withdrawal planning that allows planners to improve the probability of success while simultaneously increasing the average withdrawal rate. Researchers have also used mortality and longevity risk for portfolio choice and asset allocation such as Bodie et al. (2004); Dybvig and Liu (2005); Babbel and Merrill (2007); Chen et al. (2006); Jimenez-Martin and Sanchez-Martin (2007); Milevsky and Huang (2011).

More recently, another leading practitioner Blanchett et al. (2012) studied the safe withdrawal rate and provided a framework for choosing a withdrawal strategy, initial withdrawal rate, and asset allocation. While Blanchett et al. (2012) use a utility model to compare the outcomes of various strategies, they do not prescribe or recommend any one particular withdrawal strategy to be utilized over another. On a related note, Pye (2012) also looks at the withdrawal rate and considers a different question: When to retrench as opposed to determining the optimal withdrawal strategy alluding that it may be a futile objective given that many retirees will not be able to sustain themselves in retirement given that they have not saved enough; so perhaps individuals should withdraw an amount to sustain themselves and retrench when necessary.

To fill an apparent gap in the literature, in this paper we solve the full Merton (1971) model in which investment returns as well as lifetimes are random, but one in which pension annuities are also available. Moreover, we show that in such a world – with parameters properly calibrated to real world values – it is optimal to exhaust ones financial resources before becoming a centenarian. Now, of course, this does not imply that one starves to death. Rather, if indeed the retiree reaches that age they should plan to live off their pension annuity income (if it is available). Stated bluntly, if there is only a 5% chance of reaching the age of 100, it is quite rational to (i.) assume that you won't and (ii.) reduce your consumption to the minimal pension level, if you do. As a by-product to our main qualitative claim, this paper also offers an algorithm for determining optimal drawdown or spending rates in the presence of stochastic returns, exogenous pension income and longevity risk. We demonstrate that the resulting procedure is reasonably well-approximated by a deterministic algorithm originally presented in Milevsky and Huang (2011) – so long as the calculations are repeated on a frequent basis.

1.2 Agenda of the paper

The remainder of the paper is organized as follows. In Section 2 we summarize the findings in Milevsky and Huang (2011). In Section 3 we discuss the setup of the mathematical problem, the resulting PDE and related boundary conditions. In Section 4 we present several numerical examples followed by a conclusion in Section 5. All tables and figures are placed in section 6 to ease the reading and layout. The solution methodologies are provided in technical appendix A (Appendix A.1 reproduces the equations from Milevsky and Huang (2011) using the model equations from this paper and the method of characteristics. Appendix A.2 shows the numerical scheme used to solve the PDE equations in this paper. Appendix A.3 is an approximation based on asymptotic expansion and perturbation theory).

In sum, the two central questions we raise in this paper are: (1) What happens when lifetimes and investment returns are both stochastic; and (2) Can the deterministic methodology developed in Milevsky and Huang (2011) be made to work by some suitable modifcation?

For comparison purposes we present three distinct solutions. The first one is $\alpha = 0$ case, i.e., 100% in risk-free assets (labeled *static*). The second one is an approximation whereby wealth follows the stochastic processes (for a fixed asset allocation to the risky and risk-free assets) while the optimal consumption rate is solved in a deterministic setting, computed using the portfolio return rate on a frequent basis (labeled *dynamic*). The final solution is computed using the optimal control approach as described in this paper (labeled *stochastic*).

2 Review of Spending when Returns are Deterministic

In 2011, the authors of Milevsky and Huang (2011) derived, analyzed and explained the optimal retirement spending policy for a utility maximizing agent facing a stochastic lifetime. The authors deliberately ignored market risk to focus on the role of longevity risk aversion in determining optimal consumption (or spending) during a period of stochastic length.

While the 4% withdrawal rate is consistent with life-cycle consumption smoothing, it is so only under a very limited set of implausible preference parameters. According to the authors, there is no fixed withdrawal policy – the forward-looking spending rate is proportional to survival probabilities that is adjusted upwards for pension income and downward for longevity risk aversion.

Computationally, the analytic problem maximizes the value function that is the actuarially discounted utility of consumption:

$$\max_c V(c) = \int_0^\infty e^{-\rho s} \ {}_s p_x \ u(c_s) \ ds \tag{1}$$

where x is the age of the retiree, ρ is the subjective discount rate, ${}_tp_x = \exp(-\int_0^t \lambda_{x+q} dq)$ is the survival probability for an x-year old to age $x+t$, c_t is consumption, and u is the CRRA utility function. ${}_tp_x$ is parameterized using Gompertz-Makeham law of mortality under which the biological hazard rate is $\lambda_t = \lambda_0 + \frac{1}{b} e^{(t-m)/b}$. λ_0 captures the death rate attributable to accidents, m is the modal value of life and b denotes the dispersion coefficient. The utility function exhibits constant relative risk aversion (CRRA) that is given by $u(c) = \frac{c^{1-\gamma}}{(1-\gamma)}$ with γ being the parameter of risk aversion.

The wealth trajectory, denoted W_t – the dynamic constraint in the model linked to the objective function – is expressed as:

$$dW_t = [rW_t - c_t + \pi]dt \tag{2}$$

where π is the (constant) pension income and c_t is the consumption rate.

It can be shown by the Euler-Lagrange Theorem from Calculus of Variations that the optimal trajectory for W_t, in the region over which it stays positive, follows a second order non-homogeneous differential equation:

$$\ddot{W}_t - (k_t + r)\dot{W}_t + rk_t W_t = -\pi k_t \tag{3}$$

where $k_t = \frac{r-\rho-\lambda_t}{\gamma}$. The above equation is only valid until the Wealth Depletion Time (τ) since that is the time when wealth reaches zero. The solution to the differential equation above is expressed as:

$$W_t = \left(W_0 + \frac{\pi}{r}\right) e^{rt} - a_x^t(r-k, m^*, b) c_0^* e^{rt} - \frac{\pi}{r} \tag{4}$$

where W_0 is the initial wealth, $m^* = m + b \ln \gamma$ is the modified Gompertz-Makeham modal value in the annuity factor. The actuarial present value function, denoted $a_x^T(v, m, b)$ depends upon the survival probability curve via the parameters (m and b) and the valuation rate v. It is defined and computed using the following:

$$a_x^T(v, m, b) = \int_0^T e^{-vs} \ {}_s p_x \ ds \tag{5}$$

The optimal consumption rate is shown to satisfy the following equation:

$$c_t^* = c_0^* \ e^{kt} \ {}_t p_x{}^{1/\gamma} \tag{6}$$

where $k = \frac{(r-\rho)}{\gamma}$. The initial consumption c_0^* is obtained from equation (4) using the definition of WDT ($W_\tau = 0$):

$$c_0^* = \frac{(W_0 + \pi/r)e^{r\tau} - \pi/r}{a_x^\tau(r-k, m^*, b)e^{r\tau}} \tag{7}$$

and WDT satifies the equation:

$$\frac{(W_0 + \pi/r)e^{r\tau} - \pi/r}{a_x^\tau(r-k, m^*, b)e^{r\tau}} e^{k\tau} \cdot {}_\tau p_x{}^{(1/\gamma)} = \pi \tag{8}$$

The main insights in a deterministic framework of Milevsky and Huang (2011) are as follows:

1. The initial spending rate critically depends upon a retiree's risk aversion and pre-existing pensions.
2. The optimal consumption (i.e. sum of all pensions and withdrawals from the account) is a declining function of age. Retirees should consume more today than what they consume in the future.

3. Risk aversion and survival probabilities interact in a manner whereby the greater risk averse individual behaves as if he/she will live longer and therefore spends less in retirement. However, the monotonic behaviour of consumption only occurs when $\alpha = 0$. See the results and discussion in section 4 when α is non-zero.
4. Wealth trajectory declines with age and retirees with sufficient pensions spend down their wealth well ahead of reaching an advanced age.
5. The rational reaction to portfolio shock is non-linear and depends upon pre-existing pensions.
6. Converting some of the initial investible wealth into a stream of lifetime income increases consumption at all ages even when interest rates are low. For example, in the simplest case where an individual has no pre-existing pensions, the initial consumption, obtained from equation(7), will be $\frac{W_0}{a_x^\tau(r-k,m^*,b)}$. If this client were to annuitize completely, then his annuity income would be $\frac{W_0}{a_x^\infty(r,m,b)}$ which is higher.

3 Problem Setup (Stochastic Returns)

Consider an account with wealth W_t satisfying the following dynamics:

$$dW_t = [\alpha(\mu - r) + r]W_t dt + \alpha\sigma W_t dB_t + \pi dt - c_t dt \tag{9}$$

where α is the fraction of wealth in the risky asset, π is the (constant) pension income and c_t is the consumption rate. The risky asset follows a GBM process with drift μ and volatility σ while the riskless return rate is r. α is a fixed parameter between 0 and 1. We impose $W_t \geq 0$. Observe, if $\alpha = 0$, we revert back to equation (2) and the solution as presented in section 2.

Next, we define a value function (in retirement with no bequest motive) as:

$$J(t,w) = \max_{c_s} E\left[\int_t^\infty e^{-\rho s}\ {}_sp_x\ u(c_s)ds |\, W_t = w\right], \tag{10}$$

One clear departure from Merton (1971) is that in our setup the asset allocation is fixed over the entire retirement time horizon. We let $\alpha = 0\%$, 20%, 60%, or 80% over the entire investment time horizon assuming that, while in retirement, the investor will maintain the asset mix by trading and rebalancing continuously. There is a utility loss from the inability to adjust asset allocation over time. See, for example, the work by Browne et al. (2003).

3.1 Hamilton-Jacobi-Bellman Equation

The value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$J_t + \max_{c_t}\left\{[\alpha(\mu - r) + r]wJ_w + (\pi - c_t)J_w + \frac{(\alpha\sigma w)^2}{2}J_{ww} + \exp\left[-\int_0^t(\rho + \lambda_{x+q})dq\right]u(c_t)\right\} = 0 \tag{11}$$

for $w \geq 0$. It is slightly more convenient to work with $J = \exp\left[-\int_0^t(\rho + \lambda_{x+q})dq\right]V$, where V satisfies

$$V_t + \max_{c_t}\left\{[\alpha(\mu - r) + r]wV_w + (\pi - c_t)V_w + \frac{(\alpha\sigma w)^2}{2}V_{ww} + u(c_t)\right\} = (\rho + \lambda_{x+t})V. \tag{12}$$

3.2 Boundary Conditions

To solve the HJB numerically, we must find out what happens at $w = 0$. Assuming V_w and V_{ww} are bounded, then as the limit $w \to 0$ we drop wV_w and w^2V_{ww} terms to obtain:

$$V_t + (\pi - c_t^*)\,V_w + u(c_t^*) = (\rho + \lambda_{x+t})V. \tag{13}$$

where, c_t^* is the optimal consumption. If $\pi = 0$, we have the same setup as Merton (1971). If If $c_t^* \leq \pi$, we do not need a boundary condition at $w = 0$. This is due to the fact that the slope of the characteristics is positive (as we are solving the HJB backwards in time), i.e.,

$$\frac{dw}{dt} = \pi - c_t^* \geq 0. \tag{14}$$

On the other hand, if the value given by the first-order condition satisfies $c_t^* > \pi$, we set $c_t^* = \pi$ (as there are no assets to deplete). The HJB on the boundary $w = 0$ is now an ODE

$$V_t + u(c_t^*) = (\rho + \lambda_{x+t})V, \tag{15}$$

which can be integrated in time (backwards).

3.3 Wealth Depletion Age

When the return is deterministic it is well known, in the Yaari (1965) model with pensions, that there exists a wealth depletion age. This was first discovered or pointed out by Leung (1994). When the return is stochastic there is no longer a fixed wealth depletion age. One indicator is the time t when c_t^* computed by the first-order condition first exceeds π on the boundary $w = 0$. We define $T_d(t, w)$ to be the expected wealth depletion time, conditioned on $W_t = w$. It is given by solving the following partial differential equation

$$\frac{\partial T_d}{\partial t} + c_1 \frac{\partial T_d}{\partial w} + \frac{c_2^2}{2} \frac{\partial^2 T_d}{\partial w^2} + 1 = 0 \tag{16}$$

where

$$c_1 = (\alpha(\mu - r) + r))w + \pi - c_t^*, \ c_2 = \alpha\sigma w.$$

The boundary condition is needed at $w = 0$ when only $c_t^* \geq \pi$, which is $T_d(t, 0) = 0$. The expected wealth depletion age, can be easily computed by adding the starting age to T_d.

4 Numerical Examples

In this section we provide a variety of numerical examples under different parameter values and conditions to "prove" our assertion that the deterministic approximation provides satisfactory results relative to the "true" optimal control policy.

4.1 Parameters

To generate our results we need capital market assumptions (r, μ, σ), mortality assumptions (λ_0, m, b), and a measure of risk aversion (γ). Our capital market assumptions are based on the 2016 Long-Term Capital Market Assumptions from J.P. Morgan Asset Management. Specifically, we use the returns assumption of US Large Cap as a proxy for risky assets, i.e. $\mu = 8\%$ and $\sigma = 14\%$, and US Short Duration Government Treasury as a proxy for risk-free return, i.e. $r = 3.75\%$. We fit the parameters of the Gompertz-Makeham model (namely λ_0, m, b) to the RP2014 table from the Society of Actuaries with projection scale table MP2014 applied for 2015. For males, the Gompertz-Makeham parameters are $\lambda_0 = 0.003069$, $m = 89.1$, $b = 8.6$, and for female, the mortality parameters are $\lambda_0 = 0.001978$, $m = 91.2$, and $b = 8.5$. We assume that the subjective discount rate ρ equals to the risk-free rate r.

A variety of studies have estimated the value of γ. One of the earliest papers is the work by Friend and Blume (1975), which has withstood the test of time and provides an empirical justification for constant relative risk aversion, estimates the value of γ to be between 1 and 2. Feldstein and Ranguelova (2001); Mitchel et al. (1999) in the economics literature have employed values of less than 3. Mankiw and Zeldas (1991); Blake and Burrows (2001); Campbell and Viceira (2002) suggest that risk aversion levels might be higher. On the other hand, to avoid the problem of picking a γ value, Browne et al. (2003) invert the Merton optimum to solve for γ. However, any formulaic approach requires that we have the client's complete financial balance sheet inclusive of financial and real assets. Therefore, in the absence of a complete balance sheet, we present our results for several values of γ ranging from 0.5 to 6 (for the interested, results obtained by implying γ from the Merton optimum are presented in Appendix B).

4.2 No Pension Annuities

In Table 1, we present the optimal spending rate as a function of age (65, 70 and 75) and asset allocation (0%, 20%, 60% and 80%) without any pension income. We note that since the asset return is stochastic, we can no longer be certain about the wealth level at a future time, except for $\alpha = 0$ case. Instead of starting with an initial wealth at the age of 65, we compute the optimal spending rate with the same initial wealth of $100 (thousands) at age 65, 70, and 75. As is expected, the initial spending rate goes up when we increase the starting age. We also tabulate results from the approximation described in Appendix A.3. The results obtained using the method in A.3 and the numerical method are different (the difference being less than 2%) due to approximation errors.

To see the impact of age for the same retiree with a starting wealth of $100 (thousands) at age 65, we have plotted one realization of the wealth dynamics and corresponding spending rate in Figure 1. It can be seen that both wealth dynamics and spending rate under stochastic investment return deviate from those under deterministic (risk-free) return. On the other hand, the general conclusion of a declining wealth and spending rate over time remains valid. Furthermore, the approximate solution based on the deterministic approach, whereby the rate is adjusted to the return of the portfolio and solved annually with an updated wealth level, agrees well with that of the full optimal control solution.

4.3 With Pension Annuities

In Table 2 and Table 3, we present the optimal spending rate as a function of risk aversion and pension levels for an aggressive, balanced, and conservative asset allocation (80%, 60%, and 20%) for a 65 year old male and 75 year old male, respectively. We also include the value of the expected wealth depletion age. While the initial optimal spending rate (across static, dynamic, and stochastic methods) does not vary significantly, the *realized* spending rate over the lifecycle is no longer smooth and varies over a wide range for $\alpha = 60\%$ and even more for $\alpha = 80\%$, as shown in Figure 2.

When $\alpha = 0$, i.e. deterministic returns, $\rho = r$, and π is held constant, one expects consumption to decline as risk aversion (γ) rises. However, when returns are stochastic, this phenomenon isn't quite true. For example, in the case of a 65-year old male, Table 2, we observe consumption rising and later falling with an increase in γ.

Similarly, if α and γ are held constant, we observe that with increasing levels of pensions, a retiree ought to withdraw more from the account thus depleting the account early. But once again, we find that for a younger client and low risk aversion level this observation does not quite hold. For example, in Table 2, for $\gamma = 0.5$, the initial consumption upon retirement is \$19.609 and \$23.658 for pension levels $\pi = \$15$ and $\pi = \$20$, respectively, while the withdrawal rate is \$4.609 and \$3.658. Observe that regardless of the spending rate going down, total consumption still rises if pensions are higher.

To better understand what may be at work, let's take a look at equation (A20). Initial consumption of a retiree is not only a function of the growth rate, discount rate, and risk aversion but it is also a function of the mortality rate. Such a phenonmenon can also be observed in Milevsky and Huang (2011) and the initial consumption's dependence upon growth rate, discount rate, risk aversion, and mortality can also be observed in equation (7). Risk aversion (γ) plays a critical role in the calculation of optimal consumption. While an increase in γ causes an increase in the modal value of life, it also plays a role in determining the effective valuation rate $(r - k)$ of the actuarial present value function. In a deterministic setting, since $\rho = r$, means that $k = 0$. The initial consumption thus declines as γ increases. See the results (Table 4) of the fully stochastic HJB solution when $\alpha = 0$. However, in the non-deterministic case, r is replaced by r_e, the effective return rate, which implies that k is no longer zero. Depending upon the value of α and γ, the effective valuation rate $(r - k)$ of the actuarial present value function in equation (7) along with the modified modal value (m^*) may lead to a non-monotonic behaviour in initial consumption as γ increases.

As in the case without pension annuities, the optimal spending strategy derived in Milevsky and Huang (2011) for the deterministic case is quite close to the optimum when adapted to the wealth fluctuation, despite the fact that it was derived based on a deterministic investment returns.

We also observe that while pension levels have an impact on the optimal spending strategy, optimal spending appears to be more sensitive to changes in asset allocation. Changes in asset allocation implies higher volatility of asset returns. When we increase volatility ($\sigma = 25\%$) we observe not only greater volatility in consumption but also differences between the solutions from numerical methods and the approximations. See the bottom two panels in Figure 2.

4.4 Expected Wealth Depletion Age

In Figure 3, we plot the expected wealth depletion age as a function of initial wealth for a 65-year old male with a normalized pension of \$12 invested in a balanced portfolio. We observe that ruin occurs earlier for less initial wealth. We also plot the expected wealth depletion age as a function of risk aversion (γ). Once again, wealth depletion occurs later for more risk averse individuals.

In Table 5 we provide the Wealth Depletion Times (WDT) for a 65 year old male, for a \$100 investment at $\pi = 4, 8, 12, 15, 20$ (per \$100 wealth) and $\gamma = 0.5, 1, 2, 4$, and 6. WDT is solved both numerically and using Monte Carlo Simulations. Observe that as pension levels increase WDT tends to decrease. On the other hand and as expected, WDT rises with increasing risk aversion.

It is also important to note, that wealth depletion time is shortened only in the context of higher pension levels. In other words, consumption is not zero but it is equal to pension when wealth gets depleted. Clearly if a retiree does not have any pensions we see that wealth is depleted at the terminal age; on the other hand, for those retirees with high levels of pension income relative to their investible wealth, wealth is depleted much earlier than the terminal age.

5 Conclusion

In this paper we develop the optimal solution to the retirement spending problem from the world of pensions and retirement income planning. In a practitioner-oriented paper with an emphasis on "ease of use", Milevsky and Huang (2011) developed an optimal retirement spending policy for a utility-maximizing retiree facing a stochastic lifetime but assuming deterministic investment returns. Of course, in the real world financial returns are unknown, it is natural to question the relevance of Milevsky and Huang (2011) in a world of stochastic investment returns.

In this paper we solve the retirement spending problem when investment returns are stochastic using the well-established principles of stochastic control theory and dynamic programming. We impose the non-borrowing constraint and assume that wealth is non-negative which differs from the classical Merton (1971) solution. After solving the full problem properly, we compare with the above-mentioned approximation solution. Our approximate solution is obtained by computing the withdrawal rate using the solution under deterministic return on a yearly basis, while allowing the wealth process to be stochastic, thus adapting to current market conditions, when the model is parameterized to realistic (historical) equity and bond return coefficients.

In other words, the simplistic approximation – when calibrated properly and frequently – can indeed be used as an accurate guide for retirement spending policy.

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6 Figures and Tables

Table 1: Optimal retirement spending (in dollars) and expected wealth depletion age for an initial \$100 investment, as a function of age and asset allocation. Assumes no pension income: $\pi = 0$, and risk aversion $\gamma = 2$.

	Numerical Method Results				Approximation based on A.3			
Age	$\alpha = 0\%$	$\alpha = 20\%$	$\alpha = 60\%$	$\alpha = 80\%$	$\alpha = 0\%$	$\alpha = 20\%$	$\alpha = 60\%$	$\alpha = 80\%$
65	\$6.278	\$6.561	\$6.988	\$7.105	\$6.365	\$6.676	\$7.322	\$7.653
70	\$7.051	\$7.328	\$7.755	\$7.870	\$7.161	\$7.469	\$8.105	\$8.430
75	\$8.148	\$8.418	\$8.851	\$8.965	\$8.291	\$8.598	\$9.229	\$9.551

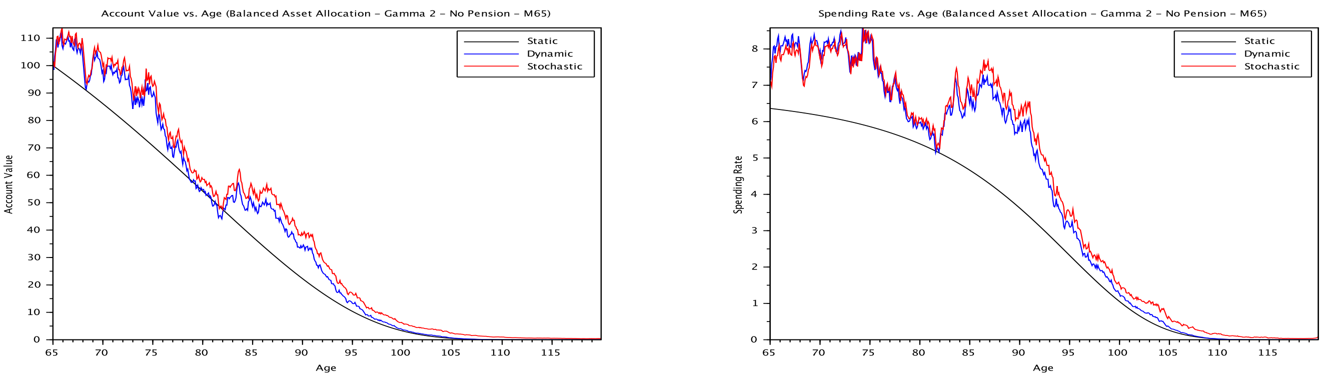

Figure 1: Wealth dynamics (left panel) and optimal spending rate (right panel) for one realization, computed using both the approximate method and the optimal control approach for $\alpha = 60\%$ and $\gamma = 2$ for a 65 year old male. For comparison purposes, the solution for $\alpha = 0\%$ is also plotted.

Table 2a: Per $100 Spending Rate & Ruin Age for an **Aggressive** Asset Allocation at **Age 65**.

Stock Exposure $\alpha = 80\%$					
Pension in $	Risk Aversion (γ)				
	0.5	1	2	4	6
4	$6.516	$7.443	$7.564	$7.092	$6.574
	89.7	93.1	97.4	103.2	107.5
8	$5.874	$7.268	$7.688	$7.461	$7.091
	88.0	90.8	94.5	99.2	102.8
12	$5.164	$7.002	$7.688	$7.648	$7.375
	87.1	89.6	92.8	97.0	100.1
15	$4.609	$6.771	$7.646	$7.728	$7.515
	86.6	88.9	91.9	95.9	98.7
20	$3.658	$6.350	$7.529	$7.798	$7.670
	86.0	88.0	90.8	94.4	97.0

Table 2b: Per $100 Spending Rate & Ruin Age for a **Balanced** Asset Allocation **Age 65**.

Stock Exposure $\alpha = 60\%$					
Pension in $	Risk Aversion (γ)				
	0.5	1	2	4	6
4	$7.456	$7.656	$7.435	$6.925	$6.500
	88.4	92.2	96.7	102.5	106.6
8	$7.261	$7.717	$7.648	$7.263	$6.920
	86.5	89.7	93.7	98.5	102.0
12	$6.986	$7.687	$7.754	$7.461	$7.170
	85.4	88.3	91.9	96.3	99.4
15	$6.751	$7.632	$7.797	$7.564	$7.303
	84.8	87.5	90.9	95.1	98.0
20	$6.323	$7.503	$7.826	$7.685	$7.466
	84.0	86.5	89.7	93.5	96.2

Table 2c: Per $100 Spending Rate & Ruin Age for a **Conservative** Asset Allocation **Age 65**.

Stock Exposure: $\alpha = 20\%$					
Pension in $	Risk Aversion (γ)				
	0.5	1	2	4	6
4	$9.699	$8.282	$7.242	$6.447	$6.056
	85.4	89.9	94.9	100.8	104.7
8	$10.516	$8.920	$7.746	$6.859	$6.431
	83.0	87.0	91.5	96.7	100.1
12	$11.172	$9.427	$8.142	$7.175	$6.714
	81.5	85.2	89.5	94.3	97.4
15	$11.600	$9.757	$8.398	$7.377	$6.893
	80.7	84.3	88.4	93.0	96.0
20	$12.231	$10.241	$8.772	$7.670	$7.150
	79.6	83.0	86.9	91.3	94.2

Table 3a: Per $100 Spending Rate & Ruin Age for an **Aggressive** Asset Allocation at **Age 75**.

Stock Exposure: $\alpha = 80\%$					
Pension in $	Risk Aversion (γ)				
	0.5	1	2	4	6
4	$12.717	$11.338	$10.016	$8.652	$7.778
	90.6	94.1	98.4	103.9	107.9
8	$13.361	$11.971	$10.653	$9.354	$8.556
	88.9	91.9	95.6	100.3	103.5
12	$13.835	$12.438	$11.111	$9.830	$9.065
	87.9	90.7	94.0	98.2	101.1
15	$14.125	$12.726	$11.389	$10.112	$9.361
	87.3	90.0	93.2	97.1	99.8
20	$14.525	$13.127	$11.777	$10.496	$9.756
	86.6	89.1	92.1	95.7	98.2

Table 3b: Per $100 Spending Rate & Ruin Age for a **Balanced** Asset Allocation at **Age 75**.

Stock Exposure: $\alpha = 60\%$					
Pension in $	Risk Aversion (γ)				
	0.5	1	2	4	6
4	$13.652	$11.542	$9.897	$8.502	$7.718
	89.8	93.5	97.9	103.4	107.3
8	$14.661	$12.377	$10.616	$9.176	$8.407
	88.1	91.3	95.1	99.8	103.0
12	$15.467	$13.035	$11.166	$9.665	$8.886
	87.0	90.0	93.5	97.7	100.6
15	$15.990	$13.459	$11.516	$9.968	$9.176
	86.4	89.2	92.6	96.6	99.3
20	$16.756	$14.078	$12.021	$10.397	$9.579
	85.7	88.3	91.4	95.2	97.7

Table 3c: Per $100 Spending Rate & Ruin Age for a **Conservative** Asset Allocation at **Age 75**

Stock Exposure: $\alpha = 20\%$					
Pension in $	Risk Aversion (γ)				
	0.5	1	2	4	6
4	$15.767	$12.102	$9.705	$8.058	$7.313
	88.3	92.2	96.8	102.2	105.9
8	$17.530	$13.397	$10.672	$8.801	$7.964
	86.5	89.9	93.9	98.6	101.7
12	$18.985	$14.458	$11.454	$9.390	$8.471
	85.3	88.5	92.2	96.5	99.3
15	$19.951	$15.159	$11.969	$9.774	$8.799
	84.7	87.7	91.2	95.3	98.0
20	$21.396	$16.206	$12.732	$10.340	$9.279
	83.9	86.7	90.0	93.9	96.4

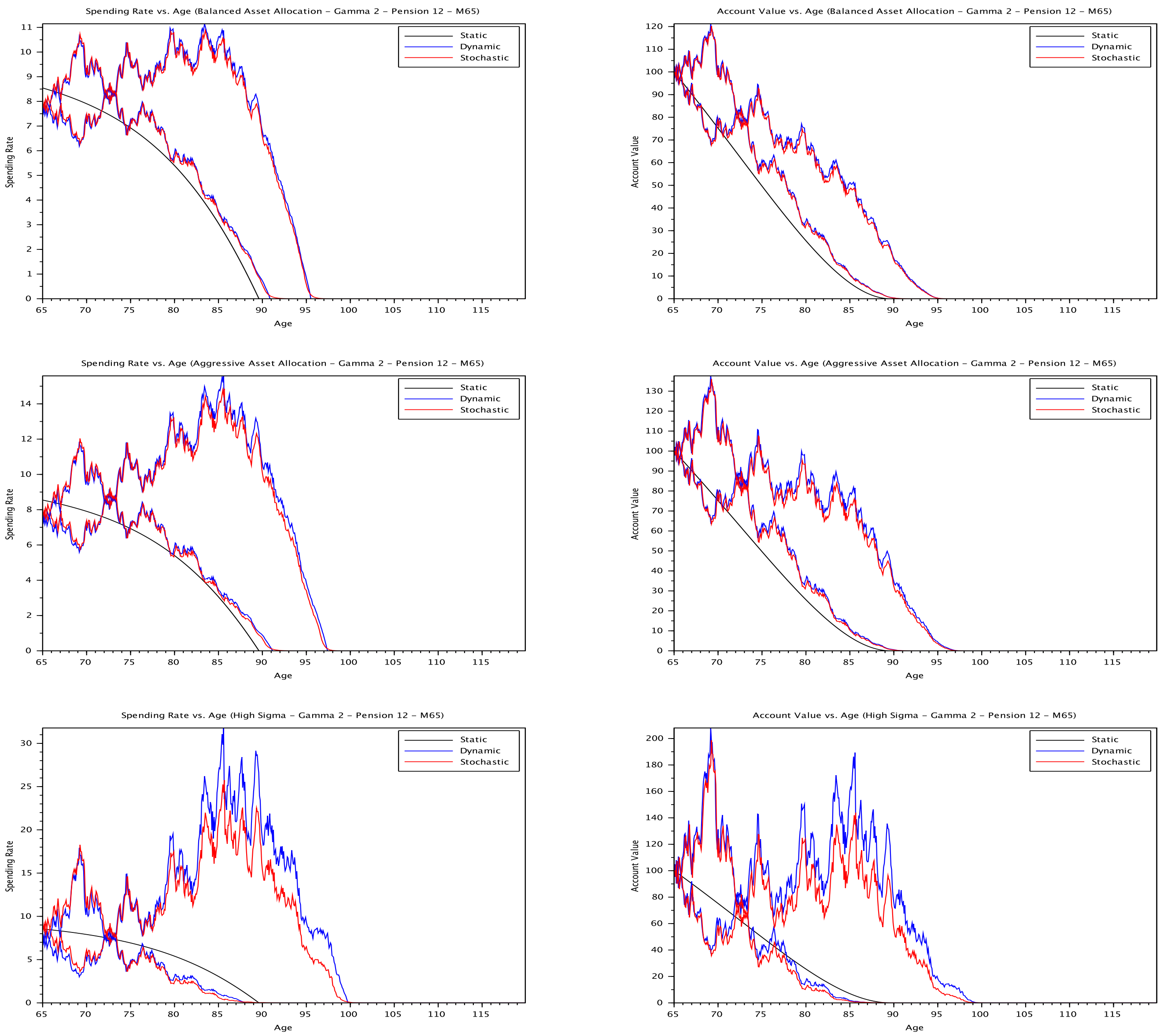

Figure 2: The optimal spending rate (left panel) and wealth dynamics (right panel) for two sample path realizations, computed using both the approximate method (blue) and the optimal control approach (red) for a 65 year old male. The exact parameters are: risky stock allocation $\alpha = 60\%$ (top), $\alpha = 80\%$ (middle); pension income $\pi = \$12$, risk aversion $\gamma = 2$. The bottom panels are examples of high volatility ($\sigma = 25\%$). For comparison purposes, the solution for $\alpha = 0\%$ is also plotted (black solid line).

Table 4: Per \$100 Spending Rate & Ruin Age – **Deterministic Returns** at **Age 65**.

Pension in \$	Risk Aversion (γ)				
	0.5	1	2	4	6
4	\$10.997	\$8.705	\$7.202	\$6.168	\$5.703
	83.8	88.5	93.8	99.8	103.7
8	\$12.335	\$9.664	\$7.901	\$6.692	\$6.154
	81.1	85.4	90.2	95.5	99.0
12	\$13.436	\$10.449	\$8.467	\$7.109	\$6.508
	79.6	83.6	88.1	93.0	96.2
15	\$14.168	\$10.969	\$8.840	\$7.382	\$6.738
	78.7	82.5	86.9	91.7	94.8
20	\$15.263	\$11.746	\$9.396	\$7.785	\$7.075
	77.6	81.2	85.3	90.0	92.9

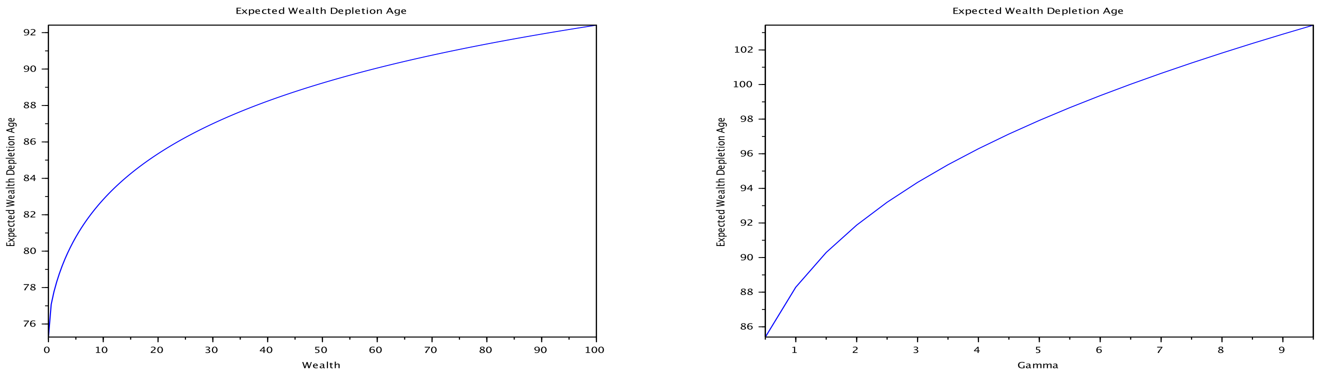

Figure 3: Expected wealth depletion age as a function of initial wealth (left panel, $\gamma = 2$) and as a function of risk aversion (right panel) for a 65-year old male invested with $\pi = \$12$ per \$100 in a balanced portfolio.

Table 5: **Age 65 Results:** At what time will the individual deplete their wealth, rationally? A comparison of numerical PDE methods and Monte Carlo simulations with standard deviations. Assuming the individual begins with \$100 and receives $\$\pi$ in pension income each year. Note that the greater the pension π, all else being equal, the earlier the Wealth Depletion Time (WDT). Also, the lower the risk aversion γ the earlier the WDT.

π		Aggressive					Balanced					Conservative				
		Risk Aversion (γ)					Risk Aversion (γ)					Risk Aversion (γ)				
		0.5	1	2	4	6	0.5	1	2	4	6	0.5	1	2	4	6
4	PDE	24.7	28.1	32.4	38.2	42.5	23.4	27.2	31.7	37.5	41.6	20.4	24.9	29.9	35.8	39.7
	MC	27.0	30.8	35.6	41.8	46.4	23.0	26.6	31.2	36.9	41.0	23.2	24.4	29.4	35.2	44.0
	SD	1.5	1.8	2.3	2.7	3.1	1.5	1.8	2.2	2.7	3.0	0.4	0.7	0.8	0.9	0.8
8	PDE	23.0	25.8	29.5	34.3	37.8	21.5	24.7	28.7	33.5	37.0	18.0	22.0	26.5	31.7	35.1
	MC	25.0	28.2	32.2	37.3	41.1	21.2	24.3	28.1	33.0	36.4	17.6	21.6	26.0	31.2	34.6
	SD	1.5	1.9	2.3	2.9	3.2	1.4	1.8	2.2	2.7	3.0	0.5	0.6	0.8	0.9	1.0
12	PDE	22.1	24.6	28.3	32.0	35.1	20.4	23.3	27.4	31.3	34.4	16.5	20.2	24.5	29.3	32.5
	MC	23.9	26.7	28.8	34.9	38.2	20.1	22.9	26.1	30.7	33.8	16.2	19.8	24.0	28.8	31.9
	SD	1.4	1.8	2.4	2.8	3.2	1.3	1.8	2.1	2.6	2.9	0.5	0.6	0.7	0.9	1.0
15	PDE	21.6	23.9	26.9	30.9	33.7	19.8	22.5	25.9	30.1	33.0	15.7	19.3	23.4	28.0	31.0
	MC	23.3	25.9	29.2	33.5	36.6	19.5	22.1	25.4	29.5	32.4	15.4	18.9	22.9	25.5	30.5
	SD	1.4	1.8	2.2	2.7	3.1	1.3	1.7	2.1	2.5	2.8	0.4	0.6	0.7	0.9	1.0
20	PDE	21.0	23.0	25.8	29.4	32.0	19.0	21.5	24.7	28.5	31.2	14.6	18.0	21.9	26.3	29.2
	MC	22.5	24.9	28.0	31.9	31.5	18.8	21.2	24.2	28.0	30.7	14.3	17.7	21.5	25.9	28.7
	SD	1.3	1.7	2.1	2.6	3.6	1.2	1.6	2.0	2.5	2.7	0.4	0.5	0.7	0.8	0.9

A Solution Methodologies

A.1 Closed Form Solution for $\alpha = 0$

We start by considering a special case where $\alpha = 0$ and the wealth dynamics is deterministic, which can be solved using the method of Calculus of Variation. Here we focus on solving the associated Hamilton–Jacobi–Bellman (HJB) equation:

$$V_t + rwV_w + (\pi - c_t^*)V_w + u(c_t^*) = (\rho + \lambda_{x+t})V \tag{A1}$$

with $c_t^* = V_w^{-1/\gamma}$ for $w \geq 0$ and $t \geq 0$.

We seek the solution of the HJB equation in the form $V(t,w) = u(\hat{w})h(t)$ where $\hat{w} = w + \beta(t)$ for two unknown functions $h(t)$ and $\beta(t)$ for $w \geq 0$. Using standard arguments, the optimal consumption rate is given by $c_t^* = \hat{w}h^{-1/\gamma}$ and the HJB becomes:

$$u(\hat{w})\left\{\dot{h} + \gamma h^{1-\frac{1}{\gamma}} + [(1-\gamma)r - \rho - \lambda_{x+t}]\,h\right\} + \hat{w}^{-\gamma}(\dot{\beta} - r\beta + \pi) = 0,$$

from which we obtain two equations for h and β as

$$\dot{h} + \gamma h^{1-\frac{1}{\gamma}} + [(1-\gamma)r - \rho - \lambda_{x+t}]\,h = 0, \tag{A2}$$

$$\dot{\beta} - r\beta + \pi = 0, \tag{A3}$$

where $\dot{h}$ and $\dot{\beta}$ denote time derivatives of h and β.

The value function V can be obtained by using the method of characteristics (for details of the method, please refer Carrier and Pearson (1976)):

$$\frac{dV}{dt} = (\rho + \lambda_{x+t})V - u(c_t^*) \tag{A4}$$

along the characteristics, or the wealth dynamics

$$\frac{dW_t}{dt} = rW_t + \pi - c_t^* \tag{A5}$$

for $W_t \geq 0$. For our purpose, we only need to solve the wealth dynamics and consumption, which is given by $c_t^* = \hat{W}_t h^{-1/\gamma}$ with $\hat{W}_t = W_t + \beta(t)$.

We now derive the differential equations satisfied by $\hat{W}_t$ and c_t^* as follows. First of all, it is a simple exercise to show that

$$\frac{d\hat{W}_t}{dt} = r\hat{W}_t - h^{-1/\gamma}\hat{W}_t + \dot{\beta} - r\beta + \pi.$$

Using the equation for β, we obtain

$$\frac{d\hat{W}_t}{dt} = r\hat{W}_t - h^{-1/\gamma}\hat{W}_t. \tag{A6}$$

A similar calculation gives us the following equation

$$\frac{dc_t^*}{dt} = \left(r - h^{-\frac{1}{\gamma}} + \frac{\dot{h}}{\gamma h}\right)c_t^*.$$

Using the equation for h, we arrive at a simplified equation for c_t^* as

$$\frac{dc_t^*}{dt} = \frac{r - \rho - \lambda_{x+t}}{\gamma}c_t^*, \tag{A7}$$

which can be integrated to yield

$$c_t^* = c_0^* \exp\left(\int_0^t k_s ds\right) \tag{A8}$$

where $k_s = (r - \rho - \lambda_{x+s})/\gamma$ and an unknown initial consumption rate c_0^*. The wealth equation can be rewritten as

$$\frac{d\hat{W}_t}{dt} = r\hat{W}_t - c_0^* \exp\left(\int_0^t k_s ds\right) \tag{A9}$$

which can be integrated as

$$W_t^* = \exp(rt)\left[\hat{W}_0 - c_0^* \int_0^t \exp\left(-ru + \int_0^u k_s ds\right)\right]. \tag{A10}$$

for $t \geq \tau$ for some unknown wealth depletion time τ. The wealth depletion time and the initial consumption rate c_0^* can be obtained using $c_\tau^* = \pi$ and $W_\tau = 0$, or

$$c_0^* = \pi \exp\left(-\int_0^\tau k_s ds\right), \tag{A11}$$

$$\hat{W}_0 = c_0^* \int_0^\tau \exp\left(-ru + \int_0^u k_s ds\right). \tag{A12}$$

These solutions are identical to the ones obtained in section 2, but using a different method.

A.2 Numerical Method for $\alpha > 0$

When a risky asset is included, i.e., $\alpha > 0$, we can solve the HJB equation numerically using a finite difference approximation. In a short-hand notation, we replace the HJB equation (12) with the following finite difference equation

$$\begin{aligned} &\frac{V_j^{(n)} - V_j^{(n-1)}}{\Delta t} + \frac{c_1 + |c_1|}{2} \frac{V_j^{(n)} - V_{j-1}^{(n)}}{\Delta w} + \frac{c_1 - |c_1|}{2} \frac{V_{j+1}^{(n)} - V_j^{(n)}}{\Delta w} \\ &= c_2 \frac{V_{j+1}^{(n)} + V_{j-1}^{(n)} - V_j^{(n)}}{(\Delta w)^2} + u(c_j^{(n-1)}) - (\rho + \lambda^{(n)}) V_j^{(n)} \end{aligned} \tag{A13}$$

for $j = 0, 1, ...J$ and $n = 0, 1, ..., N$, where

$$c_1 = c_j^{(n-1)} - \pi - (\alpha(\mu - r) + r) w_j, \tag{A14}$$

$$c_2 = \frac{(\alpha \sigma w_j)^2}{2}, \tag{A15}$$

$$c_j^{(n-1)} = \left(\frac{V_j^{(n)} - V_{j-1}^{(n)}}{\Delta w} \right)^{-\frac{1}{\gamma}}, \tag{A16}$$

and $V_j^{(n)}$ is the approximation of $V(t, w)$ on grid point (t_n, w_j). Note that we have reversed time $t_n = T - n\Delta t$, where $T = N\Delta t$ for a given N. The grid sizes Δw and Δt, and the maximum w_J are chosen to ensure that the numerical method maintains stability and that the effect of the finite domain size $0 < w < w_J$ is minimized.

The boundary condition at w_0 is given either by

$$\frac{V_0^{(n)} - V_0^{(n-1)}}{\Delta t} + \frac{c_1 - |c_1|}{2} \frac{V_1^{(n)} - V_0^{(n)}}{\Delta w} = u(c_0^{(n-1)}) - (\rho + \lambda^{(n)}) V_0^{(n)} \tag{A17}$$

when $c_0^{(n-1)} < \pi$, or

$$\frac{V_0^{(n)} - V_0^{(n-1)}}{\Delta t} = u(c_0^{(n-1)}) - (\rho + \lambda^{(n)}) V_0^{(n)} \tag{A18}$$

when $c_0^{(n-1)} > \pi$, in which case we set $c_0^{(n-1)} = \pi$ due to the no-borrowing constraint. Finally, the boundary condition at w_J and the initial condition at $t_0 = T$ are given by using the closed form Merton solution as an approximation.

Please refer to Morton and Mayer (2005) for details on the numerical solution techniques for partial differential equations.

A.3 Approximate Solution for $\alpha > 0$

When σ is small or a conservative asset allocation strategy with relatively small α, which is normally the case for retirement funds, we can use an approximate solution as an alternative to the solution of the stochastic control problem. The idea is to approximate the solution f by an asymptotic expansion $f = f^{(0)} + \sigma^2 f^{(1)} + \cdots$, where $f = V$ for the HJB equation or $f = T_d$ for the expected wealth depletion time. At the leading order, the HJB equation becomes

$$V_t^{(0)} + r_e w V_w^{(0)} + (\pi - c_t^{(0)}) V_w^{(0)} + u(c_t^{(0)}) = (\rho + \lambda_{x+t}) V^{(0)} \tag{A19}$$

with $r_e = \alpha(\mu - r) + r$ and $c_t^{(0)} = \left(V_w^{(0)} \right)^{-1/\gamma}$ for $w \geq 0$ and $t \geq 0$. Note that this is the same equation for the $\alpha = 0$ case with r replaced by r_e and the solution can be written as

$$c_t^{(0)} = c_0^{(0)} \exp \left(\int_0^t k_e ds \right) \tag{A20}$$

where $k_e = (r_e - \rho - \lambda_{x+s})/\gamma$. The approximate wealth depletion age $\tau^{(0)}$ can be once again obtained by solving the now approximate (deterministic) wealth dynamics equation

$$\frac{d\hat{W^{(0)}}_t}{dt} = r_e \hat{W^{(0)}}_t - \left(h^{(0)} \right)^{-1/\gamma} \hat{W^{(0)}}_t. \tag{A21}$$

and setting $W_{\tau^{(0)}}^{(0)} = 0$.

On the other hand, the leading order asymptotic solution of the expected wealth depletion age satisfies the following equation

$$\frac{\partial T_d^{(0)}}{\partial t} + c_1 \frac{\partial T_d^{(0)}}{\partial w} + 1 = 0 \tag{A22}$$

with $c_1 = r_e w + \pi - c_t^{(0)}$. This can be rewritten as

$$\frac{dT_d^{(0)}}{dt} = -1 \tag{A23}$$

along the path of $W_t^{(0)}$. This equation can be explicitly solved as $T_d^{(0)}(t) = -t + T_d^{(0)}(0)$. Using the fact that $T_d^{(0)}(\tau^{(0)}) = 0$ and $W_{\tau^{(0)}}^{(0)} = 0$, we obtain $T_d^{(0)}(0) = \tau^{(0)}$ and $T_d^{(0)}(t) = -t + \tau^{(0)}$. We note that in principle, the higher order correction can be obtained by obtaining and solving the HJB and the equation for the wealth depletion age, which will not be pursued here.

From a practical point of view, we can combine the stochastic wealth dynamics with the approximate solution as follows. We divide the time into smaller intervals, and over each such a time interval $[t_i,\, t_i + \Delta t]$, we apply the approximate solution based on the starting wealth W_{t_i}, and in addition to the deterministic wealth dynamics, we add back the stochastic component as

$$W_{t_{i+1}} = W_{t_i} \exp\left((r_e + \pi - c_t^{(0)} - 0.5(\alpha\sigma)^2)\Delta t + \alpha\sigma\sqrt{\Delta t} B_{t_i} \right) \tag{A24}$$

and move on to the next time interval.

B Using the Merton Optimum to Obtain Gamma

Risk aversion (γ) and allocation to risky assets (γ) are linked by the Merton optimum. In this section we use the asset allocation to solve for the risk aversion parameter. Specifically, we use the Merton optimum, $\alpha^* = \frac{1}{\gamma} \cdot \frac{(\mu - r)}{\sigma^2}$, to solve for the risk aversion parameter, γ.

Since our model allows for pension income, we do, however, modify the asset allocation to account for any pre-existing pension or annuities. We do so by calculating the present value of the pensions and aggregate it with investible wealth. Thus in our setup, the allocation to risk assets, $\alpha^* = \frac{\alpha w}{w + \bar{a}_x \pi}$, where $\bar{a}_x$ is the immediate pension annuity factor. The risk aversion parameter can be obtained using the following equation:

$$\gamma = \left[\frac{1}{\alpha} + \frac{\bar{a}_x \pi}{\alpha w}\right] \cdot \left(\frac{\mu - r}{\sigma^2}\right) \tag{B1}$$

While the above equation shows γ to be dependent upon $\bar{a}_x$ and w, and these values evolve with time, we calcualte γ at the outset and hold it constant throughout the entire decumulation horizon. The immediate pension annuity factors are calculated from the annuity market quotes from CANNEX Financial Exchanges. We took the average of top three quotes from firms with a minimum S&P credit rating of A-. Using the capital market assumptions used throughout this paper, we determine the risk adjusted equity premium to be 2.1684 and assuming that conservative, balanced, aggressive investors allocate 20%, 60%, and 80% to risky assets, respectively, we list γ values for various levels of pensions in the table below:

Table B1: CRRA Risk Aversion parameter, γ, for a male aged 65 and 75 at various investment allocations and pension levels.

Age	α	$\pi = 0$	$\pi = 4$	$\pi = 8$
65	20%	10.8	18.3	25.8
	60%	3.6	6.1	8.6
	80%	2.7	4.6	6.5
75	20%	10.8	16.3	21.7
	60%	3.6	5.4	7.2
	80%	2.7	4.1	5.4

Optimal spending rates and wealth depletion age are summarized in the table below:

Table B2: Per $100 Spending Rate & Ruin Age for male aged 65 and 75 at various investment allocations and pension levels (γ implied from Merton optimum).

Age	Pension in $	$\alpha = 20\%$	$\alpha = 60\%$	$\alpha = 80\%$
65	0	$3.543	$6.083	$6.513
		120.0	120.0	120.0
	4	$5.105	$6.478	$6.938
		117.8	106.8	104.5
	8	$5.379	$6.542	$7.008
		113.2	105.5	103.5
75	0	$3.308	$7.211	$7.919
		120.0	120.0	120.0
	4	$6.190	$8.110	$8.788
		114.4	105.6	103.8
	8	$6.308	$8.236	$8.914
		112.3	104.3	102.6

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